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Analytical solution of the equation of motion for a rigid domain wall in a magnetic material with perpendicular anisotropy

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This Rapid Communication reports the solution of the equation of motion for a domain wall in a magnetic material which exhibits high magnetocrystalline anisotropy. Starting from the Landau-Lifschitz-Gilbert equation for field-induced motion, we solve the equation to give an analytical expression, which specifies the domain-wall position as a function of time. Taking parameters from a Co/Pt multilayer system, we find good quantitative agreement between calculated and experimentally determined wall velocities and show that high-field uniform wall motion occurs when wall rigidity is assumed.

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The area of domain-wall spintronics is currently enjoying its heyday, both as a fruitful discipline for investigating how conduction electrons impart angular momentum onto lattice magnetization spins[1] and from the point of view of industrial application. Dynamical studies in domain-wall transport[2] have led to their use as memory bits,[3,4] while domain walls also play a central role in magnetic logic devices.[5] Controlling nanopillar magnetization with electron current[6] has been widely demonstrated and forms the basis for magnetic random access memory.

Many studies on domain-wall motion necessitate a full numerical treatment of the Landau-Lifschitz-Gilbert (LLG) equation, together with a description of the total magnetostatic energy. While the starting descriptions of the magnetostatic energy are well understood, the final numerical simulation often lacks the transparency of a purely analytical treatment. Domain-wall motion in Permalloy thin films is richly complicated by a variety of topological structures, which can be nearly energetically degenerate. Complications of domain-wall distortion under field include the Walker breakdown effect and, more generally, oscillatory motion, contraction, and expansion of walls—which are commensurate with the emission of spin waves. These effects are instabilities and the treatment of the wall as a singular object breaks down as the wall dissipates energy to the lattice. While Permalloy is an attractive material from the point of view of low-magnetization switching fields and low anisotropy, this type of nonlinear behavior is best avoided for reproducible shuttling of domain walls down a patterned magnetic wire. In this Rapid Communication, we focus on the description of domain-wall motion in a perpendicularly magnetized material (such as a Co/Pt multilayer). We show that, having assumed a rigid wall profile and negligible wall distortion (negligible spatial dependence of wall tilt angle), an analytical solution of the equation of motion of the wall under field comes out, and there are well-defined limits where the domain-wall motion is robustly linear. The assumption of negligible wall distortion is justified in these materials because the easy axis of the system is always perpendicular to the direction of motion. We begin with the LLG (or Gilbert) equation

$$\frac{d\mathbf{M}}{dt} = \gamma(\mathbf{M} \times \mathbf{H}_{\text{eff}}) - \frac{\alpha}{M_s}(\mathbf{M} \times \dot{\mathbf{M}}), \tag{1}$$

where γ is the gyromagnetic ratio defined as $\gamma = g\frac{\mu_B}{\hbar}$ (g is the electronic g factor and μ_B the Bohr magnetron) and α is the Gilbert damping. We write the effective magnetic field in the system as follows:

$$\mathbf{H}_{\text{eff}} = -\frac{1}{\mu_0}\frac{\delta E_d}{\delta \mathbf{m}}. \tag{2}$$

E_d is the energy density which contains the exchange, uniaxial, and magnetostatic external field energies as described by Eq. (3). In spherical coordinates it is written as

$$E_d = A[(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2] - K\cos^2\theta - \mu_0\mathbf{M}\cdot\mathbf{H}, \tag{3}$$

where K is the easy axis anisotropy constant, A is the exchange constant, and μ_0 is the magnetic permeability of free space, while θ and ϕ are the spherical polar angles of the magnetization. Here,

$$\nabla_m = \left(\frac{\partial}{\partial m}, \frac{1}{m}\frac{\partial}{\partial\theta}, \frac{1}{m\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{4}$$

The magnetization $[M = (M_x, M_y, M_z)]$ can be written in terms of the spherical polar angles (in a Cartesian vector basis) as $\mathbf{M} = M_s(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where $\phi = \phi(x,t)$ and $\theta = \theta(x,t)$ are the azimuthal and polar angles, respectively. We can write the time derivative of the magnetization in the basis vectors of spherical polar coordinates (e_m, e_θ, e_ϕ). This is a more convenient coordinate basis because the magnetic state of the system can be described by two scalar fields, representing the spherical polar angles, in the above set of equations. Further, only two coupled equations in ϕ and θ are required to describe the magnetostatics and dynamics (see, for example, Thiaville *et al.*[7]). Equation (1) now reads

$$\begin{pmatrix} \dot{M}_s \\ M_s\dot{\theta} \\ M_s \sin\theta\dot{\phi} \end{pmatrix} = \frac{\gamma}{\mu_0}\begin{pmatrix} 0 \\ \dfrac{1}{\sin\theta}\dfrac{\delta E}{\delta\phi} \\ -\dfrac{\delta E}{\delta\theta} \end{pmatrix} + \frac{\alpha}{M_s}\begin{pmatrix} 0 \\ M_s^2 \sin\theta\dot{\phi} \\ -M_s^2\dot{\theta} \end{pmatrix}. \tag{5}$$

From this vector equation, we have a system of two coupled partial differential equations, which are first order in time. We can eliminate $\dot{\phi}$ from the system of equations, and we then arrive at the following more simplified equation describing the time evolution of the magnetization angle θ:

$$\dot{\theta} = \frac{1}{M_s(1+\alpha^2)}\left[\frac{\gamma}{\mu_0}\frac{1}{\sin\theta}\frac{\delta E}{\delta\phi} - \frac{\alpha\gamma}{\mu_0}\frac{\delta E}{\delta\theta}\right]. \tag{6}$$

We calculate the effective magnetic field [Eq. (2)] by means of variational calculus in the following way:

$$\frac{\delta E_d}{\delta\theta} = \frac{\partial E_d}{\partial\theta} - \frac{d}{dx_i}\left(\frac{\partial E_d}{\partial\left(\dfrac{\partial\theta}{\partial x_i}\right)}\right), \tag{7}$$

where repeated indices are summed over and we have a similar equation for the azimuthal angle, ϕ. We now evaluate these expressions using the definition of the total magnetostatic energy from Eq. (3) and arrive at the following expressions for the variations in total energy with magnetization angle:

$$\frac{\delta E}{\delta\phi} = -2A\sin 2\theta\nabla_i\theta\nabla_i\phi - 2A\sin^2\theta\nabla_i^2\phi,$$

$$\frac{\delta E}{\delta\theta} = A\sin 2\theta(\nabla_i\phi)^2 + 2K\cos\theta\sin\theta + \mu_0 MH\sin\theta - 2A\nabla_i^2\theta.$$

These evaluated expressions are then substituted into Eq. (6),

$$\dot{\theta} = \frac{1}{M_s(1+\alpha^2)}\left[-\frac{\alpha}{\mu_0}2A\frac{\sin 2\theta}{\sin\theta}\nabla\theta\cdot\nabla\phi - \frac{\alpha}{\mu_0}2A\sin\theta\nabla^2\phi - \frac{\alpha\gamma}{\mu_0}A\sin 2\theta(\nabla\phi)^2 - \left(\frac{2\alpha\gamma}{\mu_0}K\cos\theta + \alpha\gamma M_s H\right)\sin\theta + \frac{2\alpha\gamma}{\mu_0}A\nabla^2\theta\right]. \tag{8}$$

We now write down the magnetization of the wall as a magnetostatic solution and assume that the wall is rigid and undergoes no distortion (i.e., $\nabla\phi=\mathbf{0}$ and $\nabla^2\phi=0$). The magnetization for a Bloch wall in a material with perpendicular easy axis anisotropy is taken to be

$$\mathbf{M} = M_s\left\{0, 1/\cosh\left[\tfrac{x-Q(t)}{\lambda}\right], \tanh\left[\tfrac{x-Q(t)}{\lambda}\right]\right\} = M_s(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

where x is the central coordinate of the wall magnetization and $Q(t)$ is the position of the center of the wall. We use the following parametrization for the magnetization angle θ as $\theta=\cos^{-1}\tanh(\frac{x-Q(t)}{\lambda})$ and insert this definition into the equation of motion given by Eq. (8). Taking $x=0$, we arrive at the following first-order equation for the wall position $Q(t)$:

$$-\dot{Q} = \frac{\lambda}{M_s(1+\alpha^2)}\left[-M_s H\alpha\gamma + 2\frac{\alpha\gamma}{\mu_0}\tanh\left(\frac{-Q}{\lambda}\right)\times\left(-K + \frac{A}{\lambda^2}\right)\right].$$

This equation is of the form $-\dot{Q}=A+C\tanh[-Q(t)/\lambda]$ for $x=0$. Using the substitution $u=e^{-2[-Q(t)]/\lambda}$, we arrive at the following equation:

$$-\int\frac{dt}{\lambda} = -\int\frac{du}{A+C}\left[\frac{1}{u+\beta u^2} + \frac{1}{1+\beta u}\right], \tag{9}$$

where we define the constant $\beta=(A-C)/(A+C)$. Expanding Eq. (9) by the method of partial fractions and integrating, we arrive at the following expression:

$$\int\frac{dt}{\lambda} = \frac{1}{2(A+C)}\ln\left[\frac{(1+\beta u)(u+\beta u^2)}{\beta}\right].$$

Integrating the left-hand side in time, we find a cubic equation which is implicit in u, and this can be written in the following way:

$$(1+\beta u)(u+\beta u^2) = \beta e^{2(A+C)/\lambda(t+t_0)}, \tag{10}$$

where $u=e^{-2(-Q)/\lambda}$, the constants A and C are defined below in terms of the parameters of the magnetic material, and t_0 is a constant of integration. We can solve this equation above to find the solution in the explicit form $Q(t)=F(A,C,t)$. For this, we use the method of Tartaglia and Cardano[8] for finding cubic roots, which allows us to transform the equation to a "depressed cubic" with the following substitution $u=y-2/(3\beta)$, and now

$$y^3 + \alpha_1 y - \gamma_1 = 0.$$

The constants α_1 and γ_1 are given by $(7\beta-4)/(3\beta^3)$ and $(72\beta^2-10-27\beta^2 e^{2(A+C)/\lambda(t+t_0)})/(27\beta^3)$, respectively. We now have a depressed cubic equation, and we appeal to the solution technique of Dal Ferro[9] and write the solution as $y=s-t$, where s and t are specified by the following relations:

$$3st = \alpha_1,$$

$$s^3 - t^3 = \gamma_1.$$

In order to solve the two simultaneous equations in s and t, we first substitute $s=\alpha_1/3t$ into the second equation and we get

$$t^6 + t^3\gamma_1 - \frac{\alpha_1^3}{9} = 0. \tag{11}$$

Recognizing this equation as a quadratic in t^3, we can solve to get

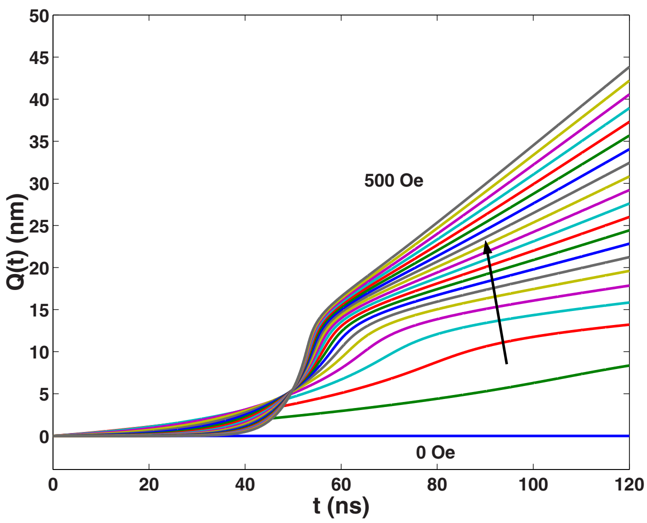

FIG. 1. (Color online) The wall position is plotted as a function of time for applied fields between 0 and 500 Oe. The Gilbert damping constant is fixed at $\alpha=0.016$. The arrow marks the increasing field.

$$t_{\pm} = \left(\frac{-\gamma_1 \pm \sqrt{\gamma_1^2 + 4\alpha_1^3/9}}{2} \right)^{1/3}. \tag{12}$$

We take the real root t_+, which corresponds to the real wall trajectory. Appealing to the definition $s=\alpha_1/3t$, we have a solution for s,

$$s = \frac{\alpha_1}{3\left[\left(-\gamma_1 + \sqrt{\gamma_1^2 + \frac{4\alpha_1^3}{9}} \right)/2 \right]^{1/3}}. \tag{13}$$

The solution for y is given by the following relation:

$$y = \frac{\alpha_1}{3[(-\gamma_1 + \sqrt{\gamma_1^2 + 4\alpha_1^3/9})/2]^{1/3}} - [(-\gamma_1 + \sqrt{\gamma_1^2 + 4\alpha_1^3/9})/2]^{1/3}.$$

Recalling that $u=y-2/(3\beta)$ and inserting this solution back into the implicit cubic in u [Eq. (10)], we find that the result of this inversion is as follows:

$$Q(t) = \frac{\lambda}{2}\ln\left(y - \frac{2}{3\beta} \right). \tag{14}$$

Recalling that the quantities α_1 and γ_1 are given by $(7\beta-4)/3\beta^3$ and $(72\beta^2-10-27\beta^2 e^{2(A+C)/\lambda(t+t_0)})/(27\beta^3)$, respectively, while $\beta=(A-C)/(A+C)$. We define the constants A and C, as follows: $A=-(\lambda\alpha\gamma H_{app})/(1+\alpha^2)$ and $C=\{\lambda 2\alpha\gamma/[M_s(1+\alpha^2)\mu_0]\}(-K+A/\lambda^2)$; and we choose the boundary condition $dQ/dt(t=0)=0$.

The results of this analytical model are plotted in Fig. 1, and we see two distinct regimes—a nonlinear region for $t<60$ ns and a linear regime, which takes over at time scales greater than 60 ns for all field values. The values used here for the calculation are taken from a Co/Pt multilayer material system[10] with perpendicular anisotropy as follows: $\alpha=0.016$, $\gamma=2.2\times10^5$ $\text{A}^{-1}\,\text{ms}^{-1}$, and $\mu_0=4\pi\times10^{-7}$ N A^{-2}; and exchange constant for Co: $A=3\times10^{-11}$ J m^{-1}, $M_s=1.5$ MA m^{-1}, $K(=K_{\text{eff}})=0.3\times10^6$ J m^{-3}, and $\lambda\sim\sqrt{A/K}=10$ nm. Note that the perpendicular anisotropy constant K here is an effective anisotropy constant, which takes into account the effect of the thin-film demagnetization field. Using these material parameters, the dynamic wall velocity ($v=dQ/dt$) versus time at various applied fields (from 0 to 500 Oe) is shown in Fig. 2(a), and this gives steady-state wall velocities in the region 0–0.5 ms^{-1}. The field direction is chosen so that reverse saturation of the magnetization occurs as the wall moves in the positive x direction. The steady-state ($t>60$ ns) wall velocity is plotted in Fig. 2(b) as a function

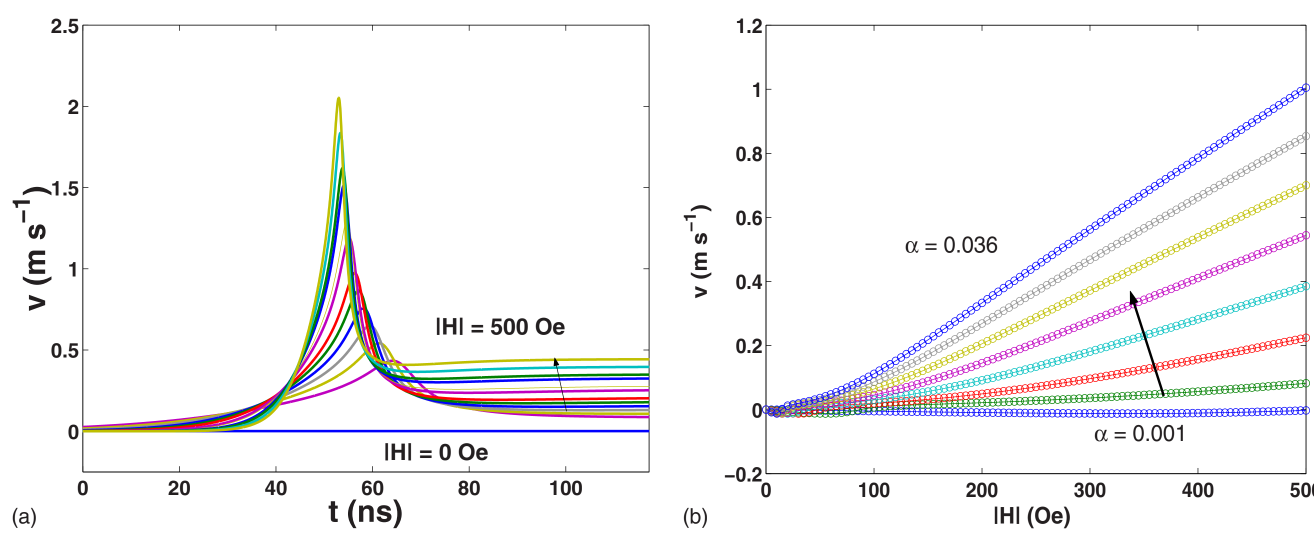

FIG. 2. (Color online) (a) Plot of the instantaneous velocity attained by the domain wall under motion by applied field at a fixed Gilbert damping parameter of $\alpha=0.016$. The plotted wall velocities here are for applied fields 0 Oe to 500 Oe; the arrow indicates the increase in field magnitude. The flat region of constant wall velocity is preceded by a critical region. (b) Plot of wall velocity as a function of field H at differing Gilbert damping constants showing the onset of wall propagation, which occurs when the critical field is reached from saturation.

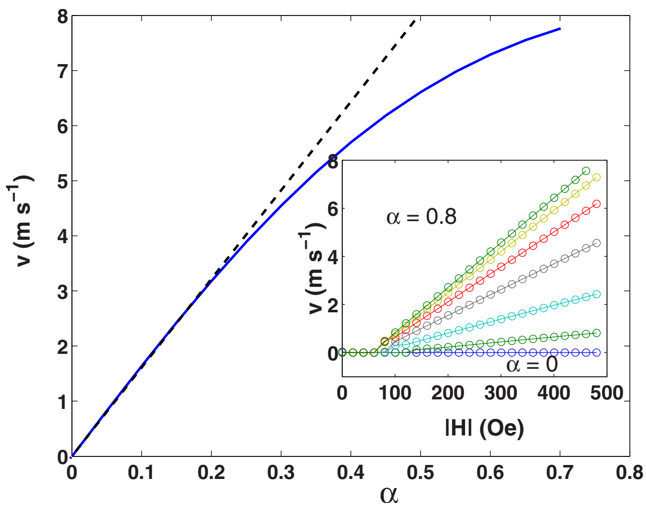

FIG. 3. (Color online) Plot of wall velocity at $|H|=500$ Oe as a function of Gilbert damping constant. The linear trend (dashed line) corresponds to the precessional regime for small α. The inset shows the field-dependent velocity at a range of Gilbert damping parameters. This calculation used magnetic parameters from the Pt/Co(0.5nm)/Pt multilayer system of Metaxas *et al.* (Ref. 14).

of applied fields at differing Gilbert damping parameters. These results show that the wall begins to move once a critical field is reached and that the wall velocity has a power-law dependence on field. Further, we plot the wall velocity in the steady-state regime at an applied field of 500 Oe against the Gilbert damping parameter α, as shown in Fig. 3. Here we find a linear relationship for small α, which corresponds to the models developed by Slonczewski[11] and others,[12,13] whereby one takes the precessional regime of steady-state wall translation (post-Walker breakdown) and writes the wall velocity as $v=\frac{\gamma\lambda}{\alpha+\alpha^{-1}}H\simeq\gamma\lambda\alpha H$, and this linear expansion is valid for small α. For $\alpha=0.3$ and at $|H|=500$ Oe, we have a wall velocity of ~ 5 ms^{-1}. This is in reasonable agreement with recently published results[14] on field-driven walls in Pt/Co(0.5 nm)/Pt thin-film systems. That work reported experimental wall velocities of $\sim 8-10$ ms^{-1} at 500 Oe with a Gilbert damping constant of about 0.3, having established anisotropy energy density, exchange stiffness, and saturated magnetization—all identical to that which we have used to parametrize our analytical model, the results of which are plotted in Fig. 3 and its inset.

This correspondence arises in the linear regime, where the wall translates uniformly and the models neglect pinning due to defects. The linear regime occurs after Walker breakdown and in the limit of a perfect wire and corresponds to the precessional regime.

In conclusion, we have calculated an analytical solution of the equation of motion for an undistorted domain wall in a perpendicularly magnetized material. This solution is constructed using first-principles arguments from energy minimum considerations, and the trajectories of the wall are completely specified by material parameters. Under the assumption of wall rigidity, we have linear wall translation above a critical threshold, where the wall position is exponentially dependent upon time. The values for wall velocities in the linear regime are in good agreement with previous experiments on field-driven walls in Pt/Co(0.5 nm)/Pt thin films, and the wall velocity is linearly dependent upon Gilbert damping corresponding to precessional motion for small Gilbert damping constant.

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